WEST BENGAL STATE UNIVERSITY

B.Sc. Honours 5th Semester Examination, 2020, held in 2021

MTMACOR12T-MATHEMATICS (CC12)

Time Allotted: 2 Hours | Full Marks: 50

The figures in the margin indicate full marks.
Candidates should answer in their own words and adhere to the word limit as practicable.
All symbols are of usual significance.

Answer Question No. 1 and any *five* from the rest

1. Answer any ***five*** questions from the following: 2×5 = 10

(a) Let G be an abelian group. Show that the mapping $f : G \to G$ defined by $f(x) = x^{-1}$, for all $x \in G$ is an automorphism of the group G.

(b) Determine the order of the automorphism group Aut ($\mathbb{Z}_{15}$) of the additive group $\mathbb{Z}_{15}$ of integers modulo 15.

(c) Prove that the subgroup $Z(G)$ (the center of a group G) is a characteristic subgroup of G.

(d) Let $G = S_3 \times \mathbb{Z}_{12}$ be the external direct product of the symmetric group S_3 of degree 3 and the additive group $\mathbb{Z}_{12}$. If $\alpha = (1\ 2\ 3) \in S_3$ and $\beta = [3] \in \mathbb{Z}_{12}$, find the order of the element (α, β) in G.

(e) Determine the number of non-isomorphic abelian groups of order 32.

(f) Examine whether a group of order 63 is simple.

(g) Let G be a group and X be a G-set. For each $x \in X$, prove that the set $G_x = \{g \in G :\ gx = x\}$ is a subgroup of the group G.

(h) Let G be a finite group and H be a subgroup of G of index $n\ (\neq 1)$ such that order of G does not divide $n!$. Prove that G contains a non-trivial normal subgroup.

(i) Let G be a finite group that has only two conjugacy classes. Show that $|G| = 2$.

(j) Let G be a finite group and H be a Sylow p-subgroup of G for some prime p. If H is a normal subgroup of G, then show that G has no Sylow p-subgroup other than H.

2. (a) Let G be a finite group of order n and m be a positive integer such that $\gcd(m, n) = 1$. Show that the mapping $\phi : G \to G$ given by $\phi(x) = x^m$, for all $x \in G$, is an automorphism of G. 4

(b) Let α be an element of the automorphism group of $\mathbb{Z}_{10}$. Then, find the possible values of $k\ (1 \leq k \leq 9)$ such that $\alpha([2]) = [k]$. 4

3. Let G be a group and N be a normal abelian subgroup of G.

(a) Show that, for each $g \in G$, the mapping $\psi_g : N \to N$ defined for all $n \in N$ by $\psi_g(n) = gng^{-1}$, is an automorphism of N. 2

(b) Prove that the mapping $\psi : G \to \mathrm{Aut}(N)$ defined by $\psi(g) = \psi_g$, for all $g \in G$ is a group homomorphism from G to $\mathrm{Aut}(N)$. 2

(c) Show that the orders of the groups $\mathrm{Aut}(N)$ and G/N are multiples of the order of $G/\ker\psi$. 2

(d) Show that $N \subseteq Z(G)$ if orders of G/N and $\mathrm{Aut}(N)$ are relatively prime. 2

4. (a) Define the commutator $[x, y]$ of two elements x and y of a group G. 1

(b) Prove that a subgroup H of a group G is a normal subgroup of G if and only if $[H, G] \subseteq H$, where $[H, G]$ denotes the subgroup generated by commutators of elements from H and from G. 2

(c) For any $\sigma \in \mathrm{Aut}(G)$, prove that $\sigma([x, y]) = [\sigma(x), \sigma(y)]$ for all $x, y \in G$. Hence, show that the commutator subgroup G' of G is characteristic in G. 3

(d) Show that G/G' is an abelian quotient group of G. 2

5. (a) (i) Let G_1 and G_2 be two finite cyclic groups. Suppose that $|G_1| = m$ and $|G_2| = n$. Prove that the external direct product $G_1 \times G_2$ of G_1 and G_2 is a cyclic group if and only if $\gcd(m, n) = 1$. 4+1

(ii) Use the result stated in (i) above, examine whether $\mathbb{Z}_8 \times \mathbb{Z}_{15} \times \mathbb{Z}_7$ is a cyclic group.

(b) Let G be a group and H, K be two subgroups of G. If G is an internal direct product of H and K, then prove that $G \simeq H \times K$. 3

6. (a) Suppose $U(n)$ denotes the group of units modulo $n > 1$. Then, for two relatively prime integers $s(>1)$ and $t(>1)$, prove that $U(st)$ is isomorphic to the external direct product $U(s) \times U(t)$ of the groups $U(s)$ and $U(t)$. 5

(b) Using the result in (a) above, prove that 1+2

(i) $U(7) \times U(15) \simeq U(21) \times U(5)$

(ii) $U(105) \simeq \mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_6$

7. (a) State the fundamental theorem of finite abelian groups. 1

(b) Let $G\ (\neq \{0\})$ be a finite abelian group and let $|G| = p_1^{n_1} p_2^{n_2}$, where p_1, p_2 are two primes and n_1, n_2 are two positive integers. Then prove that 4

(i) $G = G(p_1) \oplus G(p_2)$, and

(ii) $|G(p_i)| = p_i^{n_i}$ for each $i = 1, 2, ,$

where for any prime p, the subgroup $G(p)$ of G is given by $G(p) = \{g \in G : O(g) = p^s \text{ for some } s \geq 0\}$.

(c) Describe all the non-isomorphic abelian groups of order 504. 3

8. Let $X = \{1, 2, 3, 4, 5, 6\}$ and suppose that G is the permutation group given by the permutations of X as $\{(1),\ (1\ 2)(3\ 4\ 5\ 6), (3\ 5)(4\ 6), (1\ 2)(3\ 6\ 5\ 4)\}$. Let X be a G-set under the action given by $\sigma.x = \sigma(x)$ for all $x \in X$ and $\sigma \in G$.

(a) Find for each $\sigma \in G$, the set X_σ of fixed points of σ in X. 3

(b) Determine the stabilizer subgroups G_x of G for all $x \in X$. 3

(c) Find all distinct orbits of X under the given action. 2

9. (a) Let G be a group and X be a G-set. Suppose x, y are two elements of X having same orbit in X. Then, prove that the stabilizer subgroups G_x and G_y are isomorphic. 5

(b) Let G be a group of order 77 acting on a set X of 20 elements. Show that G must have a fixed point in X. 3

10.(a) Define permutation representation associated with a given group action. (No proof or justification is needed to show.) 1

(b) Let G be a group and A be a non-empty set. Let $\phi : G \to S(A)$ be a homomorphism from the group G to the group $S(A)$ of all permutations of the set A. Show that there is a left action of G on A, associated with which the permutation representation is the given homomorphism ϕ. 4

(c) Let G be a finite group and H be a subgroup of G of index p, where p is the smallest prime dividing the order of G. Applying generalized Cayley's theorem, show that H is a normal subgroup of G. 3

11.(a) If a group G acts on itself by conjugation, then for each $a \in G$, show that the stabilizer subgroup G_a of a in G is the centralizer $c(a)$ of a in G. 2

(b) Let p be a prime and n be a positive integer. Suppose that G be a group of order p^n. Show that $|Z(G)| > 1$. 3

(c) For any prime p, prove that every group of order p^2 is commutative. 3

12.(a) Find the class equation of S_5. 2

(b) Determine the number of distinct conjugacy classes of the symmetric group S_4. Write down the representative elements, one for each of these distinct conjugacy classes of S_4. 3

(c) Let $\sigma \in S_n$ $(n \geq 2)$ be a 3-cycle such that the order of its centralizer $c(\sigma)$ in S_n is 18. Determine the value of n and hence find the order of the conjugacy class $\text{cl}(\sigma)$ of σ in S_n. 3

13.(a) If a group G of order 68 contains a normal subgroup of order 4, show that G is a commutative group. 2

(b) By applying Sylow test for non-simplicity, show that any group of order 98 is non-simple. 2

(c) Let G be a finite group of order $p^r m$, where p is a prime number, r and m are positive integers, and p and m are relatively prime. Prove that G has a subgroup of order p^k for all, $0 \leq k \leq r$. 4

N.B. : *Students have to complete submission of their Answer Scripts through E-mail / Whatsapp to their own respective colleges on the same day / date of examination within 1 hour after end of exam. University / College authorities will not be held responsible for wrong submission (at in proper address). Students are strongly advised not to submit multiple copies of the same answer script.*

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