WEST BENGAL STATE UNIVERSITY

B.Sc. Honours 3rd Semester Examination, 2020, held in 2021

MTMACOR06T-MATHEMATICS (CC6)

Time Allotted: 2 Hours Full Marks: 50

The figures in the margin indicate full marks.
Candidates should answer in their own words and adhere to the word limit as practicable.
All symbols are of usual significance.

Answer Question No. 1 and any *five* from the rest

1. Answer any ***five*** questions from the following: 2×5 = 10

(a) Let $(S,\cdot)$ be a semigroup. If for any $x, y \in S$, $x^2 \cdot y = y = y \cdot x^2$, then prove that $(S,\cdot)$ is a group.

(b) Let G be a group such that $xy = yz \Rightarrow x = z$ for all $x, y, z \in G$. Show that G is an abelian group.

(c) Let G be a commutative group and $H = \{a \in G : O(a)$ is finite$\}$. Prove that H is a subgroup of G. [For $a \in G$, $O(a)$ stands for the order of a.]

(d) Define the center $Z(G)$ of a group G. What is the center of the Klein's 4-group K_4? Justify your answer.

(e) Determine the number of generators of a cyclic group G of order 28.

(f) Let $\alpha = (1\ 4\ 2\ 3)$ and $\beta = (1\ 3)(2\ 4)$ be two permutations in the symmetric group S_4 of degree 4. Compute the product $\alpha\beta^{-1}$ in S_4.

(g) Let $G = H \times K$ be the external direct product of two groups H and K. Prove that the set $S = \{(a, e) \in G :\ a \in H$ and e is the identity of the group $K\}$ is a normal subgroup of G.

(h) If H is a subgroup of a group G such that $x^2 \in H$, for all $x \in G$, prove that H is a normal subgroup of G.

(i) Let G be any finite group of order 70 and H be a normal subgroup of G. If H contains 14 elements, show that the factor group G/H is a commutative group.

(j) Show that there is no non-trivial homomorphism from the cyclic group $\mathbb{Z}_4$ to the cyclic group $\mathbb{Z}_7$.

2. (a) Let $a = (1\ 2)(3\ 4)$, $b = (1\ 3)(2\ 4)$ and $c = (1\ 4)(2\ 3)$ be three permutations of the set $I_4 = \{1, 2, 3, 4\}$. Suppose e denotes the identity permutation of I_4. Write down the Cayley table for the composition of permutations in $S = \{e, a, b, c\}$. Using this Cayley table, justify that S is a commutative group under the composition of permutations. [The associativity of composition of permutations may be assumed.] 5

(b) If a finite semigroup $(S, \circ)$ satisfies both sided cancellation laws, then prove that $(S, \circ)$ is a group. 3

3. (a) Examine whether the set $G=\left\{\begin{bmatrix} x & x \\ x & x \end{bmatrix}:\ x\in\mathbb{R},\ \text{with}\ x\neq 0\right\}$ forms a group under usual matrix multiplication. 4

(b) Define quasigroup. Prove that every group is a quasigroup. Is the converse true? Justify your answer. 1+2+1

4. (a) Prove that every element of a finite group is of finite order. 2

(b) Give example of an infinite group whose every element is of finite order. 2

(c) Let $(G,\circ)$ be a group and $a\in G$ be such that $a^2\circ x = x\circ a$ for some $x\in G$. Show that the order of a can never be 4. 2

(d) Determine all the elements of order 12 in the additive group $\mathbb{Z}_{36}$ of integers modulo 36. 2

5. (a) Let G be a group and H be a nonempty finite subset of G. Prove that H is a subgroup of G if and only if $ab\in H$ for all $a, b\in H$. 4

(b) Let G be a group. Then, show that the set $c(a)=\{x\in G:\ ax=xa\}$ is a subgroup of G, for every $a\in G$. Using this result, prove that the center $Z(G)$ of G is a subgroup of G. 4

6. (a) Give an example of a group in which the union of two subgroups may not be a subgroup in it. Give reasons in support of your choice of group. 1

(b) If H, K are subgroups of a group G and $HK=KH$ then show that HK is also a subgroup of G. 3

(c) Let H be a subgroup of a group G. Show that, for any $g\in G$, the set $K=\{ghg^{-1}:\ h\in H\}$ is a subgroup of G. Also show that $|K|=|H|$. 4

7. (a) Define a k-cycle on the set $I_n=\{1, 2, \ldots, n\}$. 1

(b) Let $\sigma\in S_n$ $(n\geq 2)$ be a cycle. Show that σ is a k-cycle if and only if order of σ is k in S_n. 4

(c) Find the order of the permutation α in S_8, where 3

$$\alpha=\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 7 & 2 & 6 & 5 & 3 & 1 & 4 & 8 \end{pmatrix}$$

Examine whether α is an even permutation.

8. (a) Define a cyclic group. Show that the additive group Q of rational numbers is not a cyclic group. 3

(b) Show that every subgroup of a cyclic group is cyclic. 4

(c) Find the number of subgroups of a cyclic group of order 35. 1

9. (a) Let $G=\left\{\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}:\ a, b, c \text{ are real and } ac \neq 0\right\}$ be a group under matrix multiplication. Show that $N=\left\{\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}:\ x\in\mathbb{R}\right\}$ is a normal subgroup of *G*. 3

(b) Let *G* be a finite group and *N* be a normal subgroup of *G*. Suppose that the order of *N* is relatively prime to the index *m* of *N* in *G*. Prove that $N=\{g^m:\ g\in G\}$. 3

(c) Show that the set of all even permutations in S_n, form a normal subgroup of S_n. 2

10.(a) State Lagrange's theorem for finite groups. 1

(b) Let *p* be a prime integer and *a* be an integer such that *p* does not divide *a*. Apply Lagrange's theorem to show that $a^{p-1}\equiv 1 \pmod{p}$. 2

(c) Prove that every group of prime order is cyclic. 2

(d) Let *G* be a group. Suppose that the number of elements in *G* of order 7 is 48. Determine the number of distinct subgroups of *G* of order 7. 3

11.(a) Let *G* denote the external direct product of the groups $G_1, G_2, \ldots\ldots, G_n$ $(n\geq 2)$. Show that the center of the group *G* is the external direct product of the centers of the groups $G_1, G_2, \ldots\ldots, G_n$. 2

(b) Let *G* be a finite abelian group of order *n*. If *m* is a positive integer such that *n* is divisible by *m*, then show that *G* has a subgroup of order *m*. 4

(c) Let *N* be a normal subgroup of a group *G*, and let *T* be a subgroup of the quotient group G/N. Prove that there is a subgroup *H* in *G* such that $N\subseteq H$ and $T=H/N$. 2

12.(a) Let f be a homomorphism from a group *G* to a group G'. Then show that 2+2

(i) $f(a^{-1})=f(a)^{-1}$, for all $a\in G$ and

(ii) if $a\in G$ is such that $O(a)=n$, then $O(f(a))$ divides *n*.

(b) State and prove that First Isomorphism Theorem for Groups. 1+3

13.(a) Let *G* be the multiplicative group of complex numbers and $N=\{z\in G:\ |z|=1\}$. Show that $G/N\simeq\mathbb{R}^+$, where $\mathbb{R}^+$ is the multiplicative group of positive real numbers. 4

(b) Prove that every finite cyclic group of order *n* is isomorphic to $\mathbb{Z}_n$. 4

N.B. : *Students have to complete submission of their Answer Scripts through E-mail / Whatsapp to their own respective colleges on the same day / date of examination within 1 hour after end of exam. University / College authorities will not be held responsible for wrong submission (at in proper address). Students are strongly advised not to submit multiple copies of the same answer script.*

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